Heat-Distribution Approximation of Cylindrical Wire Carrying an Alternating Current

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Abstract: **This article presents a new approach for calculating the heat distribution of a cylindrical wire carrying an alternating current. It is an approximation method that uses the skin-depth factor to distribute the heat flow into two different directions. The main objective of this method is to develop a relatively simple heat equation to calculate the temperature in cylindrical wire without using Basel functions. First, a Fourier heat equation for direct current is shown and compared with 2D FEM simulation results. Then the approximation formula will be derived from the Fourier heat equation for the case of alternating current (AC). A 2D FEM simulation is also performed for this case to validate the results of the approximation formula. The results show that the approximation formula is very suitable for most applications.**

Keywords: **Heat distribution in cylindrical wires, Fourier equation, Basel equation, FEM simulation cylindrical wire.**

I. INTRODUCTION

To determine the temperature distribution in a wire which is flown by a current is very important for a design of an electrical system like electrical motors, cables, etc. This will be done with the *Basel* functions [1] for alternating current (AC) or with a finite element method (FEM) software. The *Basel* functions are quite complicated in the application and a software is needed for the calculation. Also, for a FEM simulation a software is needed. For a quick calculation both methods are nor very suitable, thus an approximation method is needed to calculate the temperature distribution inside a wire for alternating current. The presented approximation method can be carried out in EXCEL or in a simple calculator.

At first stage a DC case will be considered and compared with a FEM simulation. Consider a cylindrical wire with the radius *R* and with the length *l* which is flown by a current.

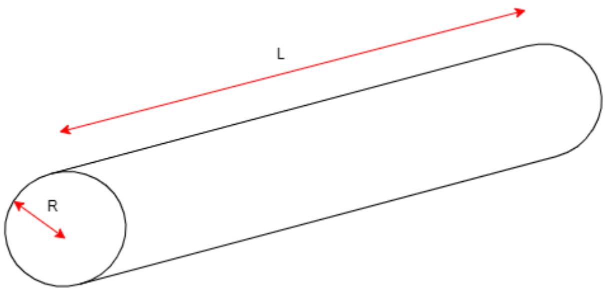

Fig. 1. Cylindrical wire, with radius *R* and length *L*, carrying a current used for temperature distribution calculation.

The analytical formula for calculating the heat distribution inside a wire through which a direct current (DC) flows is well known. It can be calculated with [2]

$$\vartheta = \begin{cases} \vartheta(r) = \vartheta_a + \frac{w}{\lambda} \cdot \frac{1}{4} \cdot (R^2 - r^2), 0 \leq r \leq R \\ \vartheta(r) = \vartheta_\infty + \dot{Q} \cdot R_{th}, \qquad r \geq R. \end{cases} \quad (1)$$

Where ϑ is the location-dependent temperature, ϑ_a the temperature on the edge of the wire, w heat source, λ thermal conductivity, R radius of the wire, ϑ_∞ the temperature in the infinity, $\dot{Q}$ rate of heat flow, and R_{th} thermal resistance (wire to air). For out calculations the following values were selected.

TABLE I. SELECTED VALUES FOR CALCULATION.

Variable	Value	Unit
λ	400	[W/(mK)]
ρ	0.0178	[Ωmm^2/m]
ϑ_∞	20	[C°]
I	10	[A]
α	6	[W/m^2K]
R	20	[mm]

The dissipation energy (joule losses) is calculated with the formula

$$w = I^2 \cdot R. \quad (2)$$

The heat flow $\vec{q}$ (represented by joule losses) spreads from the inside to the outside of the wire (fig. 2).

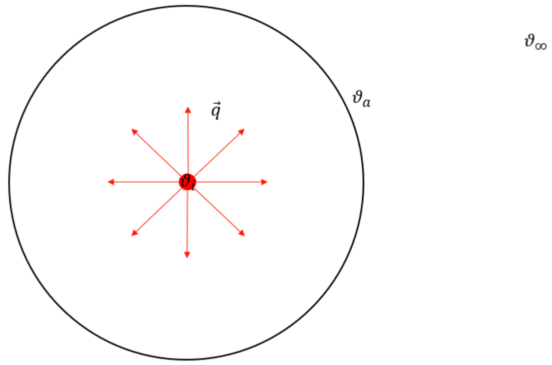

Fig. 2. Heat flow in cylindrical wire (DC), the heat flow $\vec{q}$ is representing the joule losses causing by the current und resistance of the conductor.

The surface temperature of the wire ϑ_a is influenced by the air temperature and the joule losses which can be calculated with equation (1). There is no heat source outside the wire, the air space is assumed to be infinity.

For this case a MATLAB file was written to calculate the temperature distribution of the wire. The following figures shows the results (fig. 3 and fig. 4).

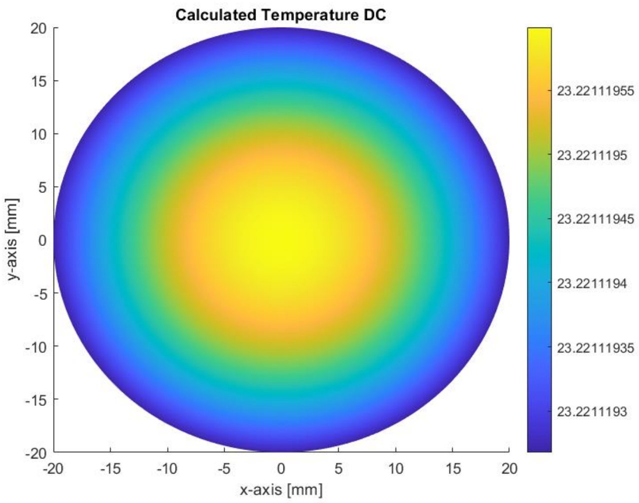

Fig. 3. Temperature distribution of cylindrical wire carrying direct current, calculated with an exact mathematical method with MATLAB, 3D illustration.

According to fig. 3 the hot spot temperature is in the middle of the conductor, furthermore the temperature is symmetrical distributed over the entire radius.

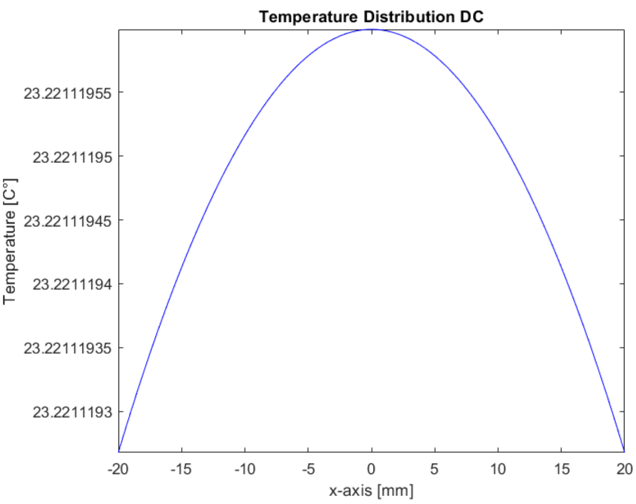

Fig. 4. Temperature distribution over radius of cylindrical wire carrying direct current, calculated with an exact mathematical method with MATLAB, 2D illustration.

Figure 4 shows the temperature over radius, there are now jumps in the behaviour, the curve is very smooth like expected from equation (1).

Also, a 2D FEM simulation was done to compare the calculated results.

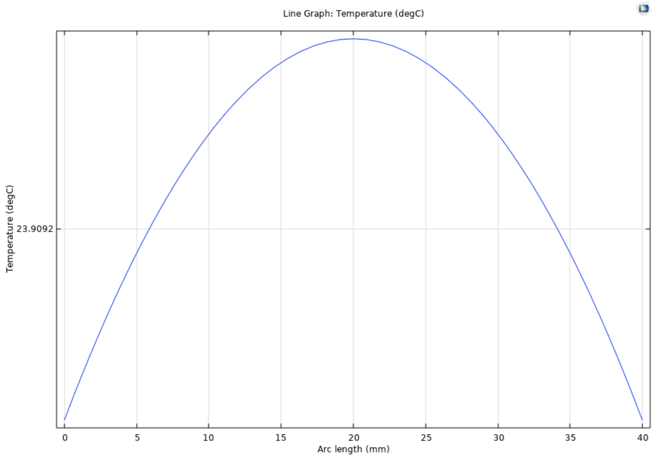

Fig. 5. Temperature distribution over radius of cylindrical wire carrying direct current. Simulation results of FEM.

The FEM simulation (Fig. 5) confirmed the calculated results. The temperature spot is like expected in the middle of the wire and falls in parabolic shape. The curve shape is almost the same, only the values are slightly different, this difference can be explained by numerical errors.

II. ANALYTICAL MODELL OF AC CASE

Now we consider the same conductor, but that one is flown by alternating current. To describe the heat distribution in an AC conductor, several preliminary assumptions regarding the current distribution have to be made. The general Fourier heat equation case is known as

$$\frac{\partial \vartheta}{\partial t} = \frac{\lambda}{\rho \cdot c} \cdot \left(\frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2} + \frac{\partial^2 \vartheta}{\partial z^2} + \frac{w}{c \cdot \rho} \right). \tag{3}$$

Since we consider here only a stationary case, the time derivative of the temperature is zero $\partial \vartheta / \partial t = 0$. So, the equation becomes

$$0 = \frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2} + \frac{\partial^2 \vartheta}{\partial z^2} + \frac{w}{\lambda}. \tag{4}$$

Furthermore, for convenience we changed the coordinate system to cylindrical coordinates

$$0 = \frac{\partial^2 \vartheta}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \vartheta}{\vartheta r} + \frac{1}{r^2} \cdot \frac{\partial^2 \vartheta}{\partial \varphi^2} + \frac{\partial^2 \vartheta}{\partial z^2} + \frac{w}{\lambda}. \tag{5}$$

We assume that the temperature along the z coordinate and the angle φ are constant

$$0 = \frac{\partial^2 \vartheta}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \vartheta}{\vartheta r} + \underbrace{\frac{1}{r^2} \cdot \frac{\partial^2 \vartheta}{\partial \varphi^2}}_{=0} + \underbrace{\frac{\partial^2 \vartheta}{\partial z^2}}_{=0} + \frac{w}{\lambda}. \tag{6}$$

This means that we do not allow heat to flow outside the wire in the z direction. Now we get

$$\frac{d^2 \vartheta}{dr^2} \cdot r + \frac{d\vartheta}{dr} + \frac{w}{\lambda} \cdot r = 0. \tag{7}$$

The differential equation (7) in this form is difficult to solve, thus the notation $\frac{d^2\vartheta}{dr^2}\cdot r+\frac{d\vartheta}{dr}$ will be summarized (product rule)

$$\frac{d^2\vartheta}{dr^2}\cdot r+\frac{d\vartheta}{dr}=\frac{d}{dr}\left(r\cdot\frac{d\vartheta}{dr}\right), \tag{8}$$

$$\frac{d}{dr}\left(r\cdot\frac{d\vartheta}{dr}\right)+\frac{w}{\lambda}\cdot r=0. \tag{9}$$

Now we solve the equation (9)

$$\frac{d}{dr}\left(r\cdot\frac{d\vartheta}{dr}\right)\cdot dr=-\frac{w}{\lambda}\cdot r\cdot dr, \tag{10}$$

$$\int\frac{d}{dr}\left(r\cdot\frac{d\vartheta}{dr}\right)\cdot dr=-\frac{w}{\lambda}\int r\cdot dr,$$

$$r\cdot\frac{d\vartheta}{dr}=-\frac{w}{\lambda}\cdot\frac{1}{2}\cdot r^2+A,$$

$$\frac{d\vartheta}{dr}=-\frac{w}{\lambda}\cdot\frac{1}{2}\cdot r+\frac{A}{r},$$

$$\int d\vartheta=-\frac{w}{\lambda}\cdot\frac{1}{2}\int r\cdot dr+A\int\frac{1}{r}\cdot dr,$$

$$\vartheta=-\frac{w}{\lambda}\cdot\frac{1}{4}\cdot r^2+A\cdot\ln(r)+B.$$

The integration constants must be determined with suitable conditions. The first condition is

$$\vartheta(R)=-\frac{w}{\lambda}\cdot\frac{1}{4}\cdot R^2+A\cdot\ln(R)+B=\vartheta_a. \tag{11}$$

This means that the surface temperature of the conductor is ϑ_a. And the second condition is

$$\frac{d\vartheta}{dr}\Big|_{r=r_i}=0. \tag{12}$$

This condition means that the derivative of the temperature, in respect to time, in $r=r_i$ is zero. The radius r_i is the location of the skin depth which penetrate the conductor. The skin depth can be calculated with

$$\delta=\sqrt{\frac{2\rho}{\omega\mu}}, \tag{13}$$

Where δ is the skin depth, ρ the specific conductivity, ω the angular frequency, and μ is the magnetic permeability. This means that the heat now flows in two directions, into the center of the wire and outside the wire (fig. 6).

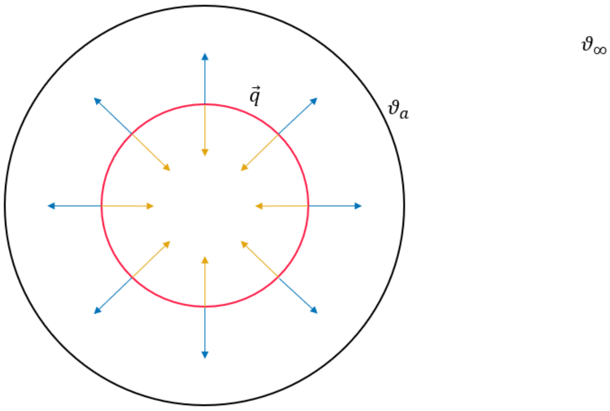

Fig. 6. Heat flow in AC case, schematic illustration. The heat flow $\vec{q}$ is located in skin depth and flows inward and outward the wire.

In figure 6 illustrate the heat flow schematic of the approximation method. It assumes that there is no current in the middle of the wire, the current density is concentrated between the radius of the wire and the skin depth.

With these assumptions an estimation formula can be derived. At first, we will determine the integration constant A

$$\frac{d\vartheta}{dr}=-\frac{w}{\lambda}\cdot\frac{1}{2}\cdot r+\frac{A}{r}.$$

So, we obtain the integration constant A

$$A=\frac{w}{\lambda}\cdot\frac{1}{2}\cdot r_i^2. \tag{14}$$

Now we will determine the integration constant B

$$\vartheta_a=-\frac{w}{\lambda}\cdot\frac{1}{4}\cdot R^2+\frac{w}{\lambda}\cdot\frac{1}{2}\cdot r_i^2\cdot\ln(R)+B$$

$$\vartheta_a=-\frac{w}{\lambda}\cdot\frac{1}{4}\cdot\left(R^2+r_i^2\cdot\ln(R)\right)+B$$

$$B=\vartheta_a+\frac{w}{\lambda}\cdot\frac{1}{4}\cdot R^2-\frac{w}{\lambda}\cdot\frac{1}{2}\cdot r_i^2\cdot\ln(R). \tag{15}$$

Now we summarize all results in one formula.

$$\vartheta(r)=-\frac{w}{\lambda}\cdot\frac{1}{4}\cdot r^2+\frac{w}{\lambda}\cdot\frac{1}{2}\cdot r_i^2\cdot\ln(r)+\vartheta_a+\frac{w}{\lambda}\cdot\frac{1}{4}\cdot R^2-\frac{w}{\lambda}\cdot\frac{1}{2}\cdot r_i^2\cdot\ln(R) \tag{16}$$

$$\vartheta(r)=-\frac{w}{\lambda}\cdot\frac{1}{4}\left(r^2-R^2+r_i^2\cdot(\ln(R)-\ln(r))\right)+\vartheta_a$$

$$\vartheta(r)=-\frac{w}{\lambda}\cdot\frac{1}{4}\left(r^2-R^2+r_i^2\cdot\ln\left(\frac{R}{r}\right)\right)+\vartheta_a. \tag{17}$$

The formula (17) is valid in $r\in[R;r_i]$. At next we have to calculate the temperature for $r\in[r_i;0]$. Since we already know the temperature at $r=r_i$

$$\vartheta(r = r_i) = -\frac{w}{\lambda} \cdot \frac{1}{4}\left(r_i^2 - R^2 + r_i^2 \cdot \ln\left(\frac{R}{r_i}\right)\right) + \vartheta_a.$$

First condition

$$\frac{d\vartheta}{dr}\Big|_{r=0} = 0$$
$$-\frac{w}{\lambda} \cdot \frac{1}{2} \cdot 0 + \frac{A}{0} = 0.$$

The temperature can't be infinity; thus, A must be zero. Now we have for the temperature in the skin depth

$$\vartheta_i = \frac{w}{\lambda} \cdot \frac{1}{4} \cdot r_i^2 + B. \tag{18}$$

So, we obtain the integration constant B

$$B = -\frac{w}{\lambda} \cdot \frac{1}{4} \cdot r_i^2 + \vartheta_i. \tag{19}$$

For the area $r \in [0; r_i]$

$$\vartheta = \frac{w}{\lambda} \cdot \frac{1}{4}(r^2 - r_i^2) + \vartheta_i. \tag{20}$$

And for the entire wire we obtain three equations

$$\vartheta(r) = -\frac{w}{\lambda} \cdot \frac{1}{4}\left(r^2 - R^2 + r_i^2 \cdot \ln\left(\frac{R}{r}\right)\right) + \vartheta_a, \tag{21}$$

for $r_i \leq r \leq R$. The second equation

$$\frac{w}{\lambda} \cdot \frac{1}{4}(r^2 - r_i^2) + \vartheta_i, \tag{22}$$

for $0 \leq r \leq r_i$. And the third equation

$$\vartheta(r) = \vartheta_\infty + \dot{Q} \cdot R_{th}, \tag{23}$$

for $r \geq R$. These three equations can be easily implemented in any suitable program.

A wire was chosen with $I_{eff} = 30\ A$, $f = 50\ kHz$, and $R = 1\ mm$. With these parameters the current distribution is shown on next figures (fig. 7 and fig. 8).

Figure 7 shows the current density calculated with *Basel* functions; this is an exact solution. Figure 8 shown the current density over the radius of the conductor. The current is mostly distributed in the outer shell of the conductor, thus the assumption that was made for the approximation method is valid when the current density is displaced in the outer shell. This is mostly the case for higher frequencies.

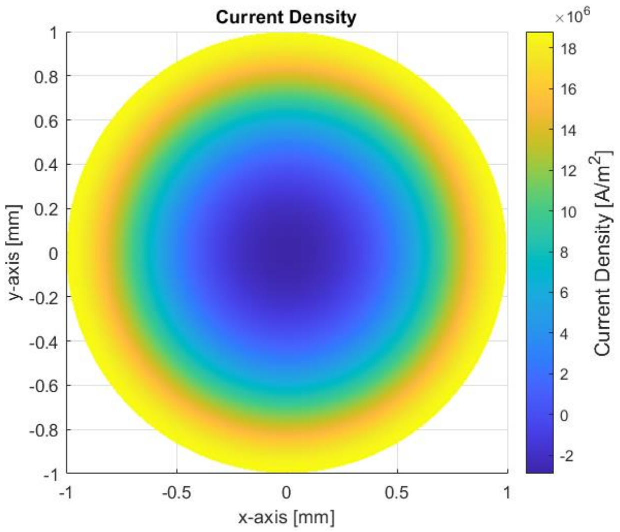

Fig. 7. Current density inside the conductor, $f = 50\ kHz$, $I = 30\ A$, and $R = 1\ mm$. 3D plot.

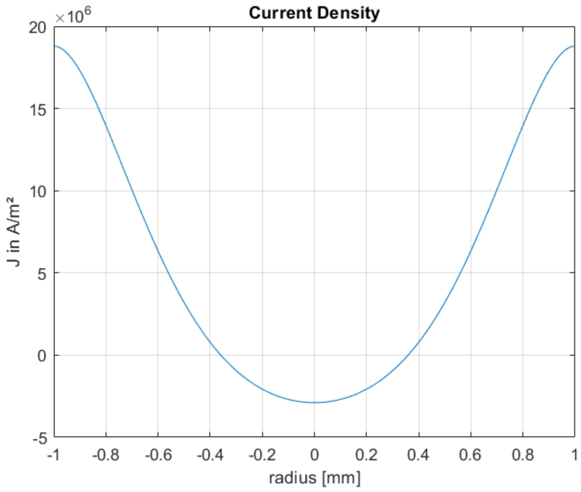

Fig. 8. Current density plot inside the conductor over radius, $f = 50\ kHz$, $I = 30\ A$, and $R = 1\ mm$. 2D plot.

Both figures show that the current is concentrated on the outside of the wire. Thus, the temperature in the middle of the wire must be almost constant because there is no heat source. The next two figures show the results of the approximation method (eq. 21, 22, and 23).

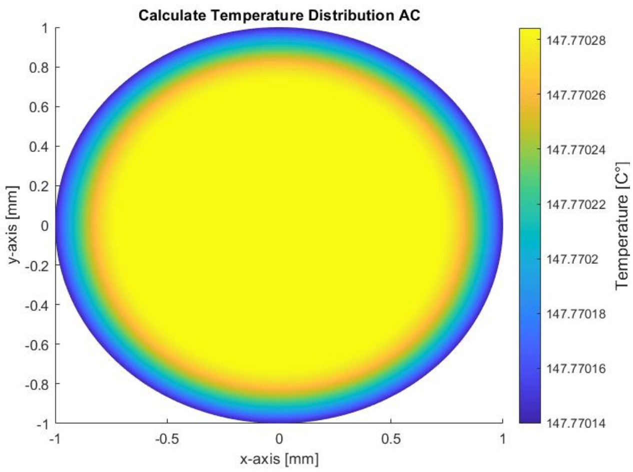

Fig. 9. Temperature distribution, approximation method 3D illustration

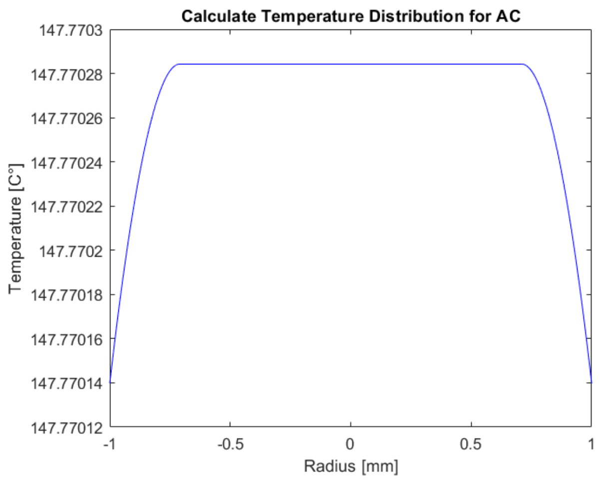

Fig. 10. Temperature distribution, approximation method 2D illustration

In the figures 9 and 10 the temperature distribution is like expected from the approximation equations (eq. 21, 22, and 23). The temperature is almost flat in the middle of the wire due to a constant derivative. According to fig. 8 there is some current density on the middle, but this will be neglected in the approximation method. After the skin depth the temperature is falling in the conductor.

Also, a FEM simulation was done with the same parameters to compare the approximation method with the exact solution. The next figure shows the result (Fig. 11).

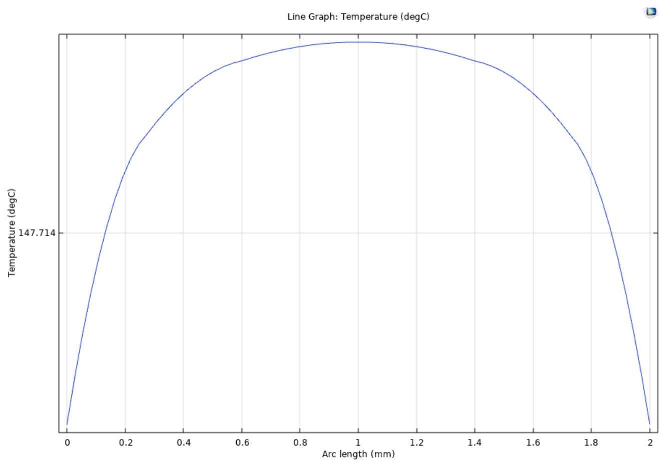

Fig. 11. Temperature distribution (alternating current) over radius, simulation with FEM.

The figure 11 (FEM simulation) shows that the temperature in the middle of the wire is not exactly flat, it's due to a current distribution in the middle of the wire. But the hot spot temperature is almost the same as in the approximating method. Also the shape (behaviour) is similar to the approximated method.

III. CONCLUSIONS

The temperature contribution for direct current in a cylindrical conductor is well known and easy to calculate with any program as shown in this publication. The hot spot temperature is in the middle of the conductor and decreased parabolic until the end of the conductor. The current distribution is constant over the entire conductor, thus the heat flow $\vec{q}$ (joule losses) is also constant in the conductor. The FEM simulations confirmed the results of the exact solution.

For alternating current the temperature contribution inside a wire can be calculated with complicated *Basel* function (fig. 7) with high accuracy. With this distribution the temperature distribution can be calculated with high effort. With the assumptions that the temperature derivative is zero at the skin depth, an approximated solution can be derived. This method includes three equations; all these equations are very simple to solve. The results of the approximation showed that the temperature distribution in the middle of the conductor is constant until the skin depth location, and fall after the skin depth location. The FEM simulation confirmed this behaviour for high frequency cases, where the current density is pushed to the outer shell of the conductor. The accuracy of the approximation method is higher for high frequency cases.

The approximation method as shown in this publication is easy to implement in any program. Furthermore, the comparison with FEM simulation showed an acceptable compromise of the approximation method.

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